Gluing methods in almost-Kähler geometry

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Objective

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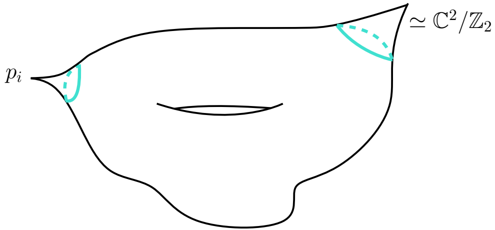
Question: Can we adapt the gluing methods developed by Arezzo and Pacard to the case where X is not a resolution of the singularities of M, but a complex deformation of one ?

Building blocks: Orbifold

Let M be a compact Kähler orbifold of complex dimension 2 with isolated singularities of type ${}^{\mathbb{C}^2}\!/_{\mathbb{Z}_2}$:

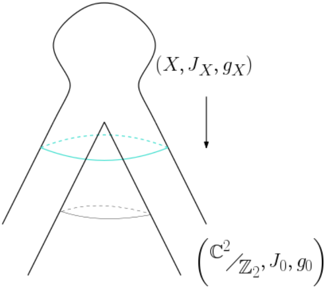


(M, ω_M, J_M, g_M)

Assume that (M, g_M) has constant scalar curvature, and that there are no nontrivial holomorphic vector fields on M.

Building blocks: ALE manifold

On the other hand let X be an ALE Kähler manifold, with zero scalar curvature and asymptotic to ${\mathbb{C}^2}/{\mathbb{Z}_2}$:



with J_X, g_X verifying, in ALE coordinates:

$$\partial^k(J_X - J_0) = O(|x|^{-4-k})$$
$$\partial^k(g_X - g_0) = O(|x|^{-4-k})$$

Example: The Stenzel structure on T^*S^2:

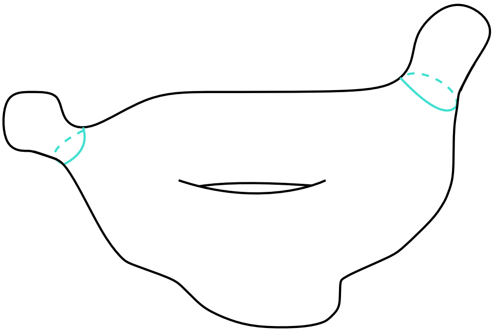
$$J_S \frac{\partial}{\partial r} = -\frac{2r}{\sqrt{r^4 - 4}} X_3,$$

$$J_S X_1 = -\sqrt{1 - \frac{4}{r^4}} X_2,$$

$$g_S = \left(1 - \frac{4}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4}\left(1 - \frac{4}{r^4}\right)\alpha_1^2 + \frac{r^2}{4}(\alpha_2^2 + \alpha_3^2)$$

Generalized connected sum

The 'connected' sum M_ε is constructed by choosing a gluing parameter $\varepsilon \ll 1$ and replacing an r_ε-neighborhood of each p_i by a suitably scaled-down 'ball' of radius $\frac{r_\varepsilon}{\varepsilon}$ in X.



M_ε

We want

- to endow M_ε with an almost-Kähler metric
- to perturb this metric into a canonical one.

Problem 1: The complex structures on X and M do not coincide. We cannot perform the connected sum in holomorphic charts !

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- On the orbifold M: equivariant Darboux theorem.
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$\Rightarrow$ M_ε is naturally a symplectic manifold $(M_\varepsilon, \omega_\varepsilon)$.

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Theorem

Let (V, ω) be a symplectic manifold. The action by conjugation of the 'Lie group' $\mathcal{G}_\omega$ of automorphisms of TM that preserve ω on $\mathcal{AC}_\omega$ is transitive. In particular, given J_1 and J_2 in $\mathcal{AC}_\omega$, there is an a in its 'Lie algebra' $\mathcal{L}_\omega$ such that

$$J_2 = \exp(a) J_1 \exp(-a);$$

moreover, the section A is unique if we assume it anticommutes with J_1 and J_2.

Cutoff on the orbifold: J_M and J_0 are both compatible with ω_M in the Darboux charts. Thus there is an endomorphism a such that

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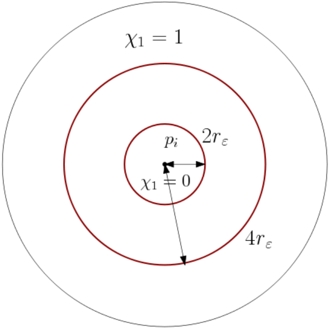

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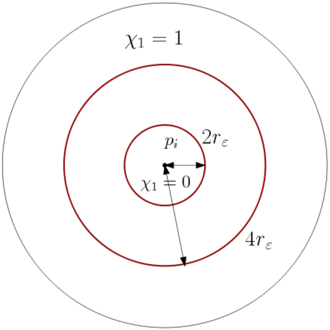

Set:

$$J_{r_\varepsilon} := \exp(\chi_1 a) J_0 \exp(-\chi_1 a).$$

Cutoff on the ALE manifold: Similarly, J_X and J_0 are compatible with ω_X outside a compact set, so we have

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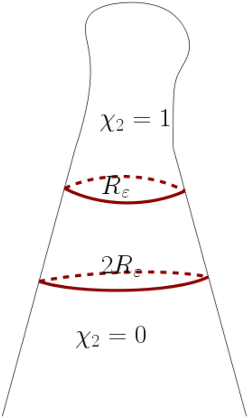

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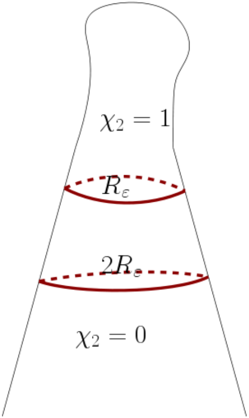
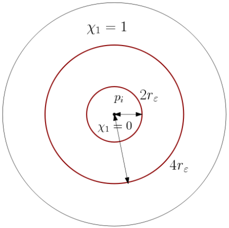


Set:

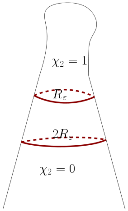
$$J_{R_\varepsilon} = \exp(\chi_2 b) J_0 \exp(-\chi_2 b).$$

We get J_ε on M_ε by identifying

the region $\{r = 2r_\varepsilon\}$ on M,



the region $\{r = 2R_\varepsilon\}$ on X



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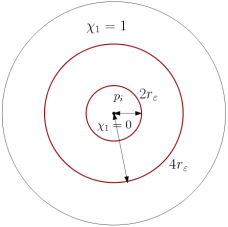

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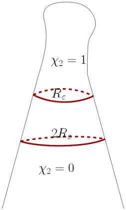

$\Rightarrow$ $(M_\varepsilon, J_\varepsilon, \omega_\varepsilon)$ is an almost-Kähler manifold.

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Solution: To f we associate the Hamiltonian vector field X_f, which induces $\mathcal{L}_{X_f} J_\varepsilon \in \mathcal{L}_{\omega_\varepsilon}$. Then we set

$$J_f = \exp(-\mathcal{L}_{X_f} J_\varepsilon)\, J_\varepsilon\, \exp(\mathcal{L}_{X_f} J_\varepsilon).$$

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Solution: The Riemannian scalar curvature does not retain the nice properties it has on a Kähler manifold.

Instead, we work with the *Hermitian scalar curvature* s^{∇}, which is the trace of the curvature of the Chern connection on the anticanonical line bundle $K^*_{J_f}$;

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Thus we want to solve

$$s^{\nabla}(J_f) = s(M) + \lambda, \qquad (\star)$$

for f in a suitable functional space.

Problem 5: How do we solve the equation $(\star)$?

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Theorem

Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded differentiable operator between Banach spaces. In a neighborhood of $0 \in \mathcal{B}_1$,

$$F(x) = F(0) + F'(0)x + Q(x).$$

Assume

1. $\|Q(x) - Q(y)\| \leq C\left(\|x\| + \|y\|\right)\|x - y\|$;
2. $\|F(0)\|_{\mathcal{B}_2} \ll 1$;
3. $F'(0)$ is an isomorphism with bounded right inverse.

Then the equation $F(x) = 0$ admits a unique solution in a small ball $B(0, r_0) \subset \mathcal{B}_1$.

We linearise the equation:

$$s^{\nabla}(J_f) = s^{\nabla}(J_{\varepsilon}) + L_{\varepsilon} f + N_{\varepsilon}(f);$$

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1. Ensure that $s^{\nabla}(J_\varepsilon)$ is close enough to $s(M)$,
2. Find a right inverse of the linearised operator L_ε,
3. Control the nonlinear term N_ε.

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Solution: It can be rewritten

$$L_\varepsilon f = -\Delta^2 f + 2\delta \mathrm{Ric}(\mathrm{grad}_{g_\varepsilon} f) + E_\varepsilon f$$
$$= \mathbb{L}_{M_\varepsilon} f + E_\varepsilon f,$$

where $\mathbb{L}$ is the *Lichnerowicz operator* on M_ε, and the error term E_ε is small, with coefficients comparable to the Nijenhuis tensor of J_ε.

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We find a right inverse to L_ε by gluing together right inverses of the Lichnerowicz operators on the model spaces: the punctured orbifold M^* and the ALE space X. The model operators have nice mapping properties provided we work in suitable functional spaces (namely, weighted Hölder spaces).

Further perspectives

- Higher dimensions ?

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Theorem [Hein, Radeasconu and Suvaina 2016]

If $n \geq 3$, every ALE Kähler manifold asymptotic to $\mathbb{C}^n/G$ is biholomorphic to a resolution of the isolated singularity $\mathbb{C}^n/G$.

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- Other types of singularities ?

Thank you for your attention